A tensor product of representations of UHF algebras arising from Kronecker products

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Abstract

We introduce a non-symmetric tensor product of representations of UHF algebras by using Kronecker products of matrices. We prove tensor product formulae of GNS representations by product states and show examples.

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1 Introduction

We have studied representations of operator algebras. We found non-symmetric tensor products of representations of some C*-algebras which are associated with non-cocommutative comultiplication of certain C*-bialgebras [12, 15, 16]. In this paper, we introduce a tensor product of representations of uniformly hyperfinite (=UHF) algebras by using Kronecker products of matrices. According to the tensor product, we show tensor product formulae of Gel'fand-Naĭmark-Segal (=GNS) representations by product states with respect to given factorizations of UHF algebras. In this section, we show our motivation, definitions and main theorems.

1.1 Motivation

In this subsection, we roughly explain our motivation and the background of this study. Explicit mathematical definitions will be shown after § 1.2.

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For any group G, there always exists the standard inner tensor product (or Kronecker product) of representations of G associated with the diagonal map from G to $G \times G$ [8, 20]. The tensor product is important to describe the duality of G [22]. For an algebra A, one does not know how to define an associative inner tensor product of representations of A unless A has a coassociative comultiplication. In [12], we introduced a non-symmetric tensor product among all representations of Cuntz algebras and determined tensor product formulae of all permutative representations completely, in spite of the unknown of any comultiplication of Cuntz algebras. In [13], we generalized this construction of tensor product to a system of C*-algebras and *-homomorphisms indexed by a monoid. For example, we constructed a non-symmetric tensor product of all representations of Cuntz-Krieger algebras by using Kronecker products of matrices [14, 15, 16]. About the basic idea of tensor products of representations of C*-algebras, see § 1.1 in [12].

On the other hand, UHF algebras and their representations are well studied [1, 2, 3, 4, 6, 9, 19, 23]. For example, GNS representations of product states of UHF algebras were completely classified by [4]. This class contains representations of UHF algebras of all Murray-von Neumann's types I, II, III ([5], § III.5).

Our interest is to construct a tensor product of representations of UHF algebras and compute tensor product formulae of this class of representations. Since one knows neither a cocommutative nor non-commutative comultiplication of UHF algebra, the tensor product is new if one can find it.

1.2 GNS representations of UHF algebras by product states

We recall GNS representations by product states of UHF algebras which are given as infinite tensor products of matrix algebras [2, 3, 23]. Let $\mathbf{N}_{\geq 2} \equiv \{2, 3, 4, \ldots\}$ and let $\mathbf{N}_{\geq 2}^{\infty}$ denote the set of all sequences in $\mathbf{N}_{\geq 2}$. For $1 \leq n < \infty$, let M_n denote the (finite-dimensional) C*-algebra of all $n \times n$-complex matrices. For $\boldsymbol{a} = (a_n)_{n \geq 1} \in \mathbf{N}_{\geq 2}^{\infty}$, the sequence $\{M_{a_n}\}_{n \geq 1}$ of C*-algebras defines the tensor product

$$\mathcal{A}_n(\boldsymbol{a}) \equiv \bigotimes_{j=1}^{n} M_{a_j}. \tag{1.1}$$

With respect to the embedding

$$\psi_{\boldsymbol{a}}^{(n)} : \mathcal{A}_n(\boldsymbol{a}) \ni A \mapsto A \otimes I \in \mathcal{A}_n(\boldsymbol{a}) \otimes M_{a_{n+1}} = \mathcal{A}_{n+1}(\boldsymbol{a}), \tag{1.2}$$

2

we regard $\mathcal{A}_n(\boldsymbol{a})$ as a C*-subalgebra of $\mathcal{A}_{n+1}(\boldsymbol{a})$ and let $\mathcal{A}(\boldsymbol{a})$ denote the inductive limit of the system $\{(\mathcal{A}_n(\boldsymbol{a}), \psi_{\boldsymbol{a}}^{(n)}) : n \geq 1\}$:

$$\mathcal{A}(\boldsymbol{a}) \equiv \varinjlim (\mathcal{A}_n(\boldsymbol{a}), \psi_{\boldsymbol{a}}^{(n)}). \tag{1.3}$$

By definition, $\mathcal{A}(\boldsymbol{a})$ is a UHF algebra of Glimm's type $\{a_1 \cdots a_k\}_{k \geq 1}$ which was classified by [9]. On the contrary, any UHF algebra is isomorphic to $\mathcal{A}(\boldsymbol{a})$ for some $\boldsymbol{a} \in \mathbf{N}_{\geq 2}^{\infty}$. Hence we call $\mathcal{A}(\boldsymbol{a})$ a UHF algebra in this paper.

Let $M_{n,+,1}$ denote the set of all positive elements in M_n which traces are 1. Then any state of M_n is written as $\omega_T(x) \equiv \operatorname{tr}(Tx)$ $(x \in M_n)$ for some $T \in M_{n,+,1}$ where tr denotes the trace of M_n. Let $\mathcal{T}(\boldsymbol{a}) \equiv \prod_{n \geq 1} M_{a_n,+,1}$ and let $\{E_{ij}^{(n)} : i, j = 1, \ldots, n\}$ denote the set of standard matrix units of M_n. For $\mathbf{T} = (T^{(n)})_{n \geq 1} \in \mathcal{T}(\boldsymbol{a})$, define the state $\omega_{\mathbf{T}}$ of $\mathcal{A}(\boldsymbol{a})$ by

$$\omega_{\mathbf{T}}(E_{j_1 k_1}^{(a_1)} \otimes \cdots \otimes E_{j_n k_n}^{(a_n)}) \equiv T_{k_1,j_1}^{(1)} \cdots T_{k_n,j_n}^{(n)} \tag{1.4}$$

for each $j_1, \ldots, j_n, k_1, \ldots, k_n$ and $n \geq 1$ where $T_{jk}^{(n)}$'s denote matrix elements of the matrix $T^{(n)}$. Then $\omega_{\mathbf{T}}$ coincides with the product state $\bigotimes_{n \geq 1} \omega_{T^{(n)}}$.

Theorem 1.1 *([10], Remark 11.4.16) For each $\mathbf{T} \in \mathcal{T}(\boldsymbol{a})$, the state $\omega_{\mathbf{T}}$ in (1.4) is a factor state, that is, if $(\mathcal{H}_{\mathbf{T}}, \pi_{\mathbf{T}}, x_{\mathbf{T}})$ is the GNS triplet of $\mathcal{A}(\boldsymbol{a})$ by $\omega_{\mathbf{T}}$, then $\pi_{\mathbf{T}}(\mathcal{A}(\boldsymbol{a}))''$ is a factor.*

The factor $\mathcal{M}_{\mathbf{T}} \equiv \pi_{\mathbf{T}}(\mathcal{A}(\boldsymbol{a}))''$ is called an *Araki-Woods factor (or infinite tensor product of finite dimensional type* I *(=ITPFI) factor)* [2, 3]. Properties of $\mathcal{M}_{\mathbf{T}}$ and $(\mathcal{H}_{\mathbf{T}}, \pi_{\mathbf{T}}, x_{\mathbf{T}})$ are closely studied in [2, 3, 21] and [4], respectively.

1.3 A set of isomorphisms

In this subsection, we introduce a set of isomorphisms among algebras $\mathcal{A}(\boldsymbol{a})$ in (1.3) and their tensor products. By using the set, we will define a tensor product of representations and that of states in § 1.4.

For $\boldsymbol{a} = (a_n), \boldsymbol{b} = (b_n) \in \mathbf{N}_{\geq 2}^{\infty}$, let $\boldsymbol{a} \cdot \boldsymbol{b} \equiv (a_1 b_1, a_2 b_2, \ldots) \in \mathbf{N}_{\geq 2}^{\infty}$ and define the $*$-isomorphism $\varphi_{\boldsymbol{a},\boldsymbol{b}}^{(n)}$ of $\mathcal{A}_n(\boldsymbol{a} \cdot \boldsymbol{b})$ onto $\mathcal{A}_n(\boldsymbol{a}) \otimes \mathcal{A}_n(\boldsymbol{b})$ by

$$\varphi_{\boldsymbol{a},\boldsymbol{b}}^{(n)}(E_{j_1 k_1}^{(a_1 b_1)} \otimes \cdots \otimes E_{j_n k_n}^{(a_n b_n)}) \equiv (E_{j_1',k_1'}^{(a_1)} \otimes \cdots \otimes E_{j_n',k_n'}^{(a_n)}) \otimes (E_{j_1'',k_1''}^{(b_1)} \otimes \cdots \otimes E_{j_n'',k_n''}^{(b_n)}) \tag{1.5}$$

for each $j_i, k_i \in \{1, \ldots, a_i b_i\}$, $i = 1, \ldots, n$ where $j_1', \ldots, j_n', k_1', \ldots, k_n';$ $j_1'', \ldots, j_n'', k_1'', \ldots, k_n''$ are defined as $j_i = b_i(j_i' - 1) + j_i''$, $k_i = b_i(k_i' - 1) + k_i''$

3

and $j_i', k_i' \in \{1, \ldots, a_i\}$, $j_i'', k_i'' \in \{1, \ldots, b_i\}$ for each $i = 1, \ldots, n$. For $\psi_{\boldsymbol{a}}^{(n)}$ in (1.2), we see that

$$(\psi_{\boldsymbol{a}}^{(n)} \otimes \psi_{\boldsymbol{b}}^{(n)}) \circ \varphi_{\boldsymbol{a},\boldsymbol{b}}^{(n)} = \varphi_{\boldsymbol{a},\boldsymbol{b}}^{(n+1)} \circ \psi_{\boldsymbol{a}\cdot\boldsymbol{b}}^{(n)} \tag{1.6}$$

for each $\boldsymbol{a}, \boldsymbol{b}$ and n. From $\{\varphi_{\boldsymbol{a}}^{(n)} : n \geq 1\}$, we can define the $*$-isomorphism $\varphi_{\boldsymbol{a},\boldsymbol{b}}$ of $\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b})$ onto $\mathcal{A}(\boldsymbol{a}) \otimes \mathcal{A}(\boldsymbol{b})$ such that

$$(\varphi_{\boldsymbol{a},\boldsymbol{b}})|_{\mathcal{A}_n(\boldsymbol{a}\cdot\boldsymbol{b})} = \varphi_{\boldsymbol{a},\boldsymbol{b}}^{(n)} \tag{1.7}$$

for each n where we identify $\mathcal{A}(\boldsymbol{a}) \otimes \mathcal{A}(\boldsymbol{b})$ with the inductive limit of the system $\{(\mathcal{A}_n(\boldsymbol{a}) \otimes \mathcal{A}_n(\boldsymbol{b}), \psi_{\boldsymbol{a}}^{(n)} \otimes \psi_{\boldsymbol{b}}^{(n)}) : n \geq 1\}$. Then the following holds:

$$(\varphi_{\boldsymbol{a},\boldsymbol{b}} \otimes id_{\boldsymbol{c}}) \circ \varphi_{\boldsymbol{a}\cdot\boldsymbol{b},\boldsymbol{c}} = (id_{\boldsymbol{a}} \otimes \varphi_{\boldsymbol{b},\boldsymbol{c}}) \circ \varphi_{\boldsymbol{a},\boldsymbol{b}\cdot\boldsymbol{c}} \quad (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbf{N}_{\geq 2}^{\infty}) \tag{1.8}$$

where $id_{\boldsymbol{x}}$ denotes the identity map on $\mathcal{A}(\boldsymbol{x})$ for $\boldsymbol{x} = \boldsymbol{a}, \boldsymbol{c}$. Equivalently, the following diagram is commutative:

$$
\begin{array}{ccc}
 & \xrightarrow{\varphi_{\boldsymbol{a},\boldsymbol{b}\cdot\boldsymbol{c}}} \quad \mathcal{A}(\boldsymbol{a}) \otimes \mathcal{A}(\boldsymbol{b} \cdot \boldsymbol{c}) \xrightarrow{id_{\boldsymbol{a}} \otimes \varphi_{\boldsymbol{b},\boldsymbol{c}}} & \\
\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b} \cdot \boldsymbol{c}) & & \mathcal{A}(\boldsymbol{a}) \otimes \mathcal{A}(\boldsymbol{b}) \otimes \mathcal{A}(\boldsymbol{c}). \\
 & \xrightarrow{\varphi_{\boldsymbol{a}\cdot\boldsymbol{b},\boldsymbol{c}}} \quad \mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b}) \otimes \mathcal{A}(\boldsymbol{c}) \xrightarrow{\varphi_{\boldsymbol{a},\boldsymbol{b}} \otimes id_{\boldsymbol{c}}} &
\end{array}
$$

Remark 1.2 (i) We consider the meaning of (1.5). For two matrices $A \in M_n$ and $B \in M_m$, define the matrix $A \boxtimes B \in M_{nm}$ by

$$(A \boxtimes B)_{m(i-1)+i', m(j-1)+j'} \equiv A_{ii'} B_{jj'} \tag{1.9}$$

for $i, i' \in \{1, \ldots, n\}$ and $j, j' \in \{1, \ldots, m\}$. The new matrix $A \boxtimes B$ is called the *Kronecker product* of A and B [7]. When $n = 1$, we see that

$$\varphi_{\boldsymbol{a},\boldsymbol{b}}^{(1)}(A \boxtimes B) = A \otimes B \quad (A \in M_{a_1},\ B \in M_{b_1}). \tag{1.10}$$

Hence $\varphi_{\boldsymbol{a},\boldsymbol{b}}^{(1)}$ is the inverse operation of the Kronecker product, which should be called the *Kronecker coproduct*.

(ii) It is clear that $\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b})$ and $\mathcal{A}(\boldsymbol{a}) \otimes \mathcal{A}(\boldsymbol{b})$ are isomorphic even if one does not know $\varphi_{\boldsymbol{a},\boldsymbol{b}}$. However, the choice of isomorphisms $\{\varphi_{\boldsymbol{a},\boldsymbol{b}}\}$ in (1.7), which satisfies both (1.6) and (1.8) is not trivial.

1.4 Main theorems

In this subsection, we show our main theorems.

1.4.1 Basic properties

In this subsubsection, we introduce a tensor product of representations and that of states of UHF algebras and show its basic properties. For a C*-algebra $\mathfrak{A}$, let $\mathrm{Rep}\mathfrak{A}$ and $\mathcal{S}(\mathfrak{A})$ denote the class of all representations and the set of all states of $\mathfrak{A}$, respectively. By using the set $\{\varphi_{\boldsymbol{a},\boldsymbol{b}} : \boldsymbol{a}, \boldsymbol{b} \in \mathbf{N}_{\geq 2}^{\infty}\}$ in (1.7), define the operation $\otimes_\varphi$ from $\mathrm{Rep}\mathcal{A}(\boldsymbol{a}) \times \mathrm{Rep}\mathcal{A}(\boldsymbol{b})$ to $\mathrm{Rep}\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b})$ by

$$\pi_1 \otimes_\varphi \pi_2 \equiv (\pi_1 \otimes \pi_2) \circ \varphi_{\boldsymbol{a},\boldsymbol{b}} \tag{1.11}$$

for $(\pi_1, \pi_2) \in \mathrm{Rep}\mathcal{A}(\boldsymbol{a}) \times \mathrm{Rep}\mathcal{A}(\boldsymbol{b})$. We see that if π_i and π_i' are unitarily equivalent for $i = 1, 2$, then $\pi_1 \otimes_\varphi \pi_2$ and $\pi_1' \otimes_\varphi \pi_2'$ are also unitarily equivalent. Furthermore, define the operation $\otimes_\varphi$ from $\mathcal{S}(\mathcal{A}(\boldsymbol{a})) \times \mathcal{S}(\mathcal{A}(\boldsymbol{b}))$ to $\mathcal{S}(\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b}))$ by

$$\rho_1 \otimes_\varphi \rho_2 \equiv (\rho_1 \otimes \rho_2) \circ \varphi_{\boldsymbol{a},\boldsymbol{b}} \tag{1.12}$$

for $(\rho_1, \rho_2) \in \mathcal{S}(\mathcal{A}(\boldsymbol{a})) \times \mathcal{S}(\mathcal{A}(\boldsymbol{b}))$. From (1.8), we see that

$$(\pi_1 \otimes_\varphi \pi_2) \otimes_\varphi \pi_3 = \pi_1 \otimes_\varphi (\pi_2 \otimes_\varphi \pi_3), \quad (\rho_1 \otimes_\varphi \rho_2) \otimes_\varphi \rho_3 = \rho_1 \otimes_\varphi (\rho_2 \otimes_\varphi \rho_3) \tag{1.13}$$

for each $(\pi_1, \pi_2, \pi_3) \in \mathrm{Rep}\mathcal{A}(\boldsymbol{a}) \times \mathrm{Rep}\mathcal{A}(\boldsymbol{b}) \times \mathrm{Rep}\mathcal{A}(\boldsymbol{c})$ and for $(\rho_1, \rho_2, \rho_3) \in \mathcal{S}(\mathcal{A}(\boldsymbol{a})) \times \mathcal{S}(\mathcal{A}(\boldsymbol{b})) \times \mathcal{S}(\mathcal{A}(\boldsymbol{c}))$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbf{N}_{\geq 2}^{\infty}$.

The following fact is a paraphrase of well-known results of tensor products of factors (which will be proved in § 2).

Fact 1.3 *Let π_1 and π_2 be representations of $\mathcal{A}(\boldsymbol{a})$ and $\mathcal{A}(\boldsymbol{b})$, respectively. Then the following holds:*

(i) *If both π_1 and π_2 are factor representations, then so is $\pi_1 \otimes_\varphi \pi_2$.*

(ii) *The type of $\pi_1 \otimes_\varphi \pi_2$ coincides with that of $\pi_1 \otimes \pi_2$ where the type of a representation π of a C*-algebra $\mathfrak{A}$ means the type of the von Neumann algebra $\pi(\mathfrak{A})''$ ([5], Theorem III.2.5.27).*

(iii) *If both π_1 and π_2 are irreducible, then so is $\pi_1 \otimes_\varphi \pi_2$.*

By definition, the essential part of the tensor product $\otimes_\varphi$ is given by the set $\{\varphi_{\boldsymbol{a},\boldsymbol{b}}\}$ of isomorphisms in (1.5). The idea of the definition of $\{\varphi_{\boldsymbol{a},\boldsymbol{b}}\}$ is an analogy of the set of embeddings of Cuntz algebras in § 1.2 of [12]. This type of tensor product is known yet in neither operator algebras nor the purely algebraic theory of quantum groups [11].

Remark 1.4 (i) Our terminology "tensor product of representations" is different from usual sense [8]. Remark that, for $\pi, \pi^{'} \in \mathrm{Rep}\mathcal{A}(\boldsymbol{a})$, $\pi \otimes_\varphi \pi^{'} \notin \mathrm{Rep}\mathcal{A}(\boldsymbol{a})$ but $\pi \otimes_\varphi \pi^{'} \in \mathrm{Rep}\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{a})$ because $\boldsymbol{a} \cdot \boldsymbol{a} \neq \boldsymbol{a}$ for any $\boldsymbol{a} \in \mathbf{N}_{\geq 2}^\infty$.

(ii) From Fact 1.3(iii), there is no nontrivial branch of the irreducible decomposition of the tensor product of any two irreducibles. In general, such a tensor product of the other algebra is decomposed into more than one irreducible component. For example, see Theorem 1.6 of [12].

1.4.2 Tensor product formulae of GNS representations

Next, we show tensor product formulae of GNS representations of product states in Theorem 1.1 as follows.

Theorem 1.5 *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{N}_{\geq 2}^\infty$ and let $\omega_\mathbf{T}$ be as in (1.4) with the GNS representation $\pi_\mathbf{T}$.*

(i) *For each $\mathbf{T} \in \mathcal{T}(\boldsymbol{a})$ and $\mathbf{R} \in \mathcal{T}(\boldsymbol{b})$,*

$$\omega_\mathbf{T} \otimes_\varphi \omega_\mathbf{R} = \omega_{\mathbf{T} \boxtimes \mathbf{R}} \tag{1.14}$$

where $\mathbf{T} \boxtimes \mathbf{R} \in \mathcal{T}(\boldsymbol{a} \cdot \boldsymbol{b})$ is defined as

$$\mathbf{T} \boxtimes \mathbf{R} \equiv (T^{(1)} \boxtimes R^{(1)}, T^{(2)} \boxtimes R^{(2)}, T^{(3)} \boxtimes R^{(3)}, \ldots) \tag{1.15}$$

for $\mathbf{T} = (T^{(n)})$ and $\mathbf{R} = (R^{(n)})$.

(ii) *For each $\mathbf{T} = (T^{(n)}) \in \mathcal{T}(\boldsymbol{a})$ and $\mathbf{R} = (R^{(n)}) \in \mathcal{T}(\boldsymbol{b})$, $\pi_\mathbf{T} \otimes_\varphi \pi_\mathbf{R}$ is unitarily equivalent to $\pi_{\mathbf{T} \boxtimes \mathbf{R}}$.*

From Theorem 1.5, the tensor product $\otimes_\varphi$ is compatible with product states and their GNS representations. More precisely, for the following two semi-groups $(\mathcal{T}, \boxtimes)$ and $(\mathcal{S}, \otimes_\varphi)$, the map

$$\mathcal{T} \equiv \bigcup_{\boldsymbol{a} \in \mathbf{N}_{\geq 2}^\infty} \mathcal{T}(\boldsymbol{a}) \ni \mathbf{T} \mapsto \omega_\mathbf{T} \in \mathcal{S} \equiv \bigcup_{\boldsymbol{a} \in \mathbf{N}_{\geq 2}^\infty} \mathcal{S}(\mathcal{A}(\boldsymbol{a})) \tag{1.16}$$

is a semigroup homomorphism. Let $\mathsf{R}_{\boldsymbol{a}}$ denote the set of all unitary equivalence classes in $\mathrm{Rep}\mathcal{A}(\boldsymbol{a})$. Then

$$\mathcal{T} \ni \mathbf{T} \mapsto [\pi_\mathbf{T}] \in \mathsf{R} \equiv \bigcup_{\boldsymbol{a} \in \mathbf{N}_{\geq 2}^\infty} \mathsf{R}_{\boldsymbol{a}} \tag{1.17}$$

is also a semigroup homomorphism between $(\mathcal{T}, \boxtimes)$ and $(\mathsf{R}, \otimes_\varphi)$ where $[\pi]$ denotes the unitary equivalence class of a representation π.

From Theorem 1.5, the following holds.

Corollary 1.6 (i) *There exist $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{N}_{\geq 2}^{\infty}$, and states $\omega \in \mathcal{S}(\mathcal{A}(\boldsymbol{a}))$ and $\omega' \in \mathcal{S}(\mathcal{A}(\boldsymbol{b}))$ such that $\omega \otimes_\varphi \omega' \neq \omega' \otimes_\varphi \omega$.*

(ii) *There exist $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{N}_{\geq 2}^{\infty}$, and classes $[\pi] \in \mathsf{R}_{\boldsymbol{a}}$ and $[\pi'] \in \mathsf{R}_{\boldsymbol{b}}$ such that $[\pi] \otimes_\varphi [\pi'] \neq [\pi'] \otimes_\varphi [\pi]$.*

From Corollary 1.6(i) and (ii), we say that $\otimes_\varphi$ is non-symmetric. In other words, two semigroups $(\mathcal{S}, \otimes_\varphi)$ and $(\mathsf{R}, \otimes_\varphi)$ are non-commutative. These non-commutativities come from the non-commutativity of the Kronecker product of matrices.

Problem 1.7 (i) Generalize the tensor product in (1.11) to that of representations of AF algebras which are not always UHF algebras.

(ii) Reconstruct UHF algebras from $(\mathcal{S}, \otimes_\varphi)$ and $(\mathsf{R}, \otimes_\varphi)$, and show a Tatsuuma duality type theorem for UHF algebras [22].

In § 2, we will prove theorems in § 1.4. In § 3, we will show examples of Theorem 1.5.

2 Proofs of main theorems

In this section, we prove main theorems.

Proof of Fact 1.3. By the definition of $\otimes_\varphi$,

$$(\pi_1 \otimes_\varphi \pi_2)(\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b})) = (\pi_1 \otimes \pi_2)(\mathcal{A}(\boldsymbol{a}) \otimes \mathcal{A}(\boldsymbol{b})). \tag{2.1}$$

(i) Since $\pi_1 \otimes \pi_2$ is also a factor representation, the statement holds from (2.1).
(ii) By definition, the type of $\pi_1 \otimes_\varphi \pi_2$ is the type of $\{(\pi_1 \otimes_\varphi \pi_2)(\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b}))\}''$. From this and (2.1), the statement holds.
(iii) By assumption, $\pi_1 \otimes \pi_2$ is also irreducible. From this and (2.1), the statement holds. ∎

Proof of Theorem 1.5. (i) By Definition 1.4, the statement holds from direct computation.
(ii) Let $\boldsymbol{a} \in \mathbf{N}_{\geq 2}^{\infty}$. For $\mathbf{T} \in \mathcal{T}(\boldsymbol{a})$, let $(\mathcal{H}_{\mathbf{T}}, \pi_{\mathbf{T}}, \Omega_{\mathbf{T}})$ denote the GNS triplet by the state $\omega_{\mathbf{T}}$. Define the GNS map $\Lambda_{\mathbf{T}}$ [17, 18] from $\mathcal{A}(\boldsymbol{a})$ to $\mathcal{H}_{\mathbf{T}}$ by

$$\Lambda_{\mathbf{T}}(x) \equiv \pi_{\mathbf{T}}(x)\Omega_{\mathbf{T}} \quad (x \in \mathcal{A}(\mathbf{T})). \tag{2.2}$$

7

Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{N}_{\geq 2}^{\infty}$. For $\mathbf{T} \in \mathcal{T}(\boldsymbol{a})$ and $\mathbf{R} \in \mathcal{T}(\boldsymbol{b})$, define the unitary $U^{(\mathbf{T}, \mathbf{R})}$ from $\mathcal{H}_{\mathbf{T} \boxtimes \mathbf{R}}$ to $\mathcal{H}_{\mathbf{T}} \otimes \mathcal{H}_{\mathbf{R}}$ by

$$U^{(\mathbf{T}, \mathbf{R})} \Lambda_{\mathbf{T} \boxtimes \mathbf{R}}(x) \equiv (\Lambda_{\mathbf{T}} \otimes \Lambda_{\mathbf{R}})(\varphi_{\boldsymbol{a}, \boldsymbol{b}}(x)) \quad (x \in \mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b})). \qquad (2.3)$$

Since $\varphi_{\boldsymbol{a}, \boldsymbol{b}}$ is bijective, $U^{(\mathbf{T}, \mathbf{R})}$ is well-defined as a unitary, and we see that

$$U^{(\mathbf{T}, \mathbf{R})} \pi_{\mathbf{T} \boxtimes \mathbf{R}}(x)(U^{(\mathbf{T}, \mathbf{R})})^* = (\pi_{\mathbf{T}} \otimes \pi_{\mathbf{R}})(\varphi_{\boldsymbol{a}, \boldsymbol{b}}(x)) = (\pi_{\mathbf{T}} \otimes_{\varphi} \pi_{\mathbf{R}})(x) \quad (2.4)$$

for $x \in \mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b})$. Hence two representations $\pi_{\mathbf{T} \boxtimes \mathbf{R}}$ and $\pi_{\mathbf{T}} \otimes_{\varphi} \pi_{\mathbf{R}}$ are unitarily equivalent. ∎

Proof of Corollary 1.6. Let $\boldsymbol{a} = \boldsymbol{b} = (2, 2, 2, \ldots) \in \mathbf{N}_{\geq 2}^{\infty}$. Define $T, R \in M_{2, +, 1}$ by $T \equiv \operatorname{diag}(1, 0)$ and $R \equiv \operatorname{diag}(0, 1)$. Define $\mathbf{T} = (T^{(n)}) \in \mathcal{T}(\boldsymbol{a})$ and $\mathbf{R} = (R^{(n)}) \in \mathcal{T}(\boldsymbol{b})$ by $T^{(n)} = T$ and $R^{(n)} = R$ for all $n \geq 1$. Then $T^{(n)} \boxtimes R^{(n)} = T \boxtimes R = \operatorname{diag}(0, 1, 0, 0)$ and $R^{(n)} \boxtimes T^{(n)} = R \boxtimes T = \operatorname{diag}(0, 0, 1, 0)$. Let $\omega \equiv \omega_{\mathbf{T}}$ and $\omega^{'} \equiv \omega_{\mathbf{R}}$. Then

$$(\omega \otimes_{\varphi} \omega^{'})(E_{22}^{(4)} - E_{33}^{(4)}) = 1 \neq -1 = (\omega^{'} \otimes_{\varphi} \omega)(E_{22}^{(4)} - E_{33}^{(4)}) \qquad (2.5)$$

where $\{E_{ij}^{(4)}\}$ denotes the set of standard matrix units of M_4. This implies (i).

From (2.5), $\|\omega \otimes_{\varphi} \omega^{'} - \omega^{'} \otimes_{\varphi} \omega\| = 2$. From Corollary 10.3.6 of [10], GNS representations π_0 and $\pi_0^{'}$ of $\mathcal{A}(\boldsymbol{a} \cdot \boldsymbol{b})$ by $\omega \otimes_{\varphi} \omega^{'}$ and $\omega^{'} \otimes_{\varphi} \omega$ are disjoint. Especially, π_0 and $\pi_0^{'}$ are not unitarily equivalent. From Theorem 1.5(i) and (ii), π_0 and $\pi_0^{'}$ are unitarily equivalent to $\pi_{\mathbf{T}} \otimes_{\varphi} \pi_{\mathbf{R}}$ and $\pi_{\mathbf{R}} \otimes_{\varphi} \pi_{\mathbf{T}}$, respectively. Therefore

$$[\pi_{\mathbf{T}}] \otimes_{\varphi} [\pi_{\mathbf{R}}] = [\pi_0] \neq [\pi_0^{'}] = [\pi_{\mathbf{R}}] \otimes_{\varphi} [\pi_{\mathbf{T}}]. \qquad (2.6)$$

Hence (ii) holds. ∎

3 Example

In this section, we show an example of Theorem 1.5. Let $\mathcal{A}(\boldsymbol{a})$ and $\mathcal{T}(\boldsymbol{a})$ be as in § 1.2. For $n \geq 2$, let $\boldsymbol{a}(n) \equiv (n, n, n, \ldots) \in \mathbf{N}_{\geq 2}^{\infty}$ and let

$$UHF_n \equiv \mathcal{A}(\boldsymbol{a}(n)) = (M_n)^{\otimes \infty}. \qquad (3.1)$$

Then UHF_n is the UHF algebra of Glimm's type $\{n^l\}_{l \geq 1}$. We consider tensor product formulae of a small class of representations of UHF_n.

Let $\{E_{ij}^{(n)}\}$ denote the set of all standard matrix units of M_n. For $j \in \{1,\ldots,n\}$, define $F_j^{(n)} \equiv E_{jj}^{(n)}$. Then $F_j^{(n)} \in M_{n,+,1}$ for each j. For $J = (j_l)_{l \geq 1} \in \{1,\ldots,n\}^\infty$, define $\mathbf{T}(J) \equiv (F_{j_1}^{(n)}, F_{j_2}^{(n)},\ldots) \in \mathcal{T}(\boldsymbol{a}(n))$.

Proposition 3.1 *Let $\pi_{\mathbf{T}}$ be as in Theorem 1.1. Then the following holds:*

(i) *For each $J \in \{1,\ldots,n\}^\infty$, $\pi_{\mathbf{T}(J)}$ is irreducible.*

(ii) *We write $P_n[J]$ as the unitary equivalence class of $\pi_{\mathbf{T}(J)}$. Then*

$$P_n[J] \otimes_\varphi P_m[K] = P_{nm}[J \cdot K] \tag{3.2}$$

for each $J = (j_l) \in \{1,\ldots,n\}^\infty$, $K = (k_l) \in \{1,\ldots,m\}^\infty$ and $n,m \geq 2$ where $J \cdot K \in \mathcal{T}(\boldsymbol{a}(nm))$ is defined by

$$J \cdot K \equiv (m(j_1 - 1) + k_1, m(j_2 - 1) + k_2, m(j_3 - 1) + k_3, \ldots). \tag{3.3}$$

Proof. (i) Since the state $\omega_{\mathbf{T}(J)}$ is the product state of pure states, $\omega_{\mathbf{T}(J)}$ is pure. Hence the statement holds.

(ii) Recall $\boxtimes$ in (1.15). Then we can verify that $\mathbf{T}(J) \boxtimes \mathbf{T}(K) = \mathbf{T}(J \cdot K)$ for each $J \in \{1,\ldots,n\}^\infty$ and $K \in \{1,\ldots,m\}^\infty$. From this and Theorem 1.5(ii), the statement holds. ∎

From Proposition 3.1, $\mathsf{P} \equiv \bigcup_{n \geq 2}\{P_n[J] : J \in \{1,\ldots,n\}^\infty\}$ is a semigroup of unitary equivalence classes of irreducible representations with respect to the product $\otimes_\varphi$. From the proof of Corollary 1.6, $(\mathsf{P}, \otimes_\varphi)$ is also non-commutative.

The class $P_n[J]$ in Proposition 3.1(ii) coincides with the restriction of a permutative representation of the Cuntz algebra $\mathcal{O}_n$ on the UHF subalgebra of $U(1)$-gauge invariant elements in $\mathcal{O}_n$, which is called an atom [1, 6]. This class contains only type I representations of UHF_n, but relations with representations of Cuntz algebras and quantum field theory are well studied [1, 6].

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